Symmetries of the Energy-Momentum Tensor of Spherically Symmetric Lorentzian Manifolds

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Abstract

Matter collineations of spherically Symmetric Lorentzian Manifolds are considered. These are investigated when the energy-momentum tensor is non-degenerate and also when it is degenerate. We have classified spacetimes admitting higher symmetries and spacetimes admitting $SO(3)$ as the maximal isometry group. For the non-degenerate case, we obtain either *four*, *six*, *seven* or *ten* independent matter collineations in which *four* are isometries and the rest are proper. The results of the previous paper [1] are recovered as a special case. It is worth noting that we have also obtained two cases where the energy-momentum tensor is degenerate but the group of matter collineations is finite-dimensional, i.e. *four* or *ten*.

Keywords : Matter symmetries, Spherically Symmetric Lorentzian Manifolds

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1 Introduction

Since the pioneering work of Katzin, Levine, Davis and their collaborators [2]-[6], the study of symmetries played an important role in the classification of spacetimes, giving rise to many interesting results with useful applications. The theory of General Relativity (GR), described by the Einstein's field equations (EFEs), is a highly non-linear. Due to its non-linearity, it becomes difficult to find the exact solutions of the EFEs, in particular, if the metric depends on all coordinates [7]. However, this problem can be overcome to some extent if it is assumed that the spacetime has some geometric symmetry properties. These symmetry properties are given by Killing vectors (KVs) which then lead to conservation laws [8]-[10]. A large number of solutions of the EFEs with different symmetry structures have been found [9] and classified according to their properties [11].

As given by the pioneers, curvature and Ricci tensors play a significant role (in terms of curvature and Ricci collineations) in understanding the geometric structure of metrics. They have provided a detailed study of curvature and Ricci collineations in the context of the related particle and field conservation laws. For a given distribution of matter, the contribution of gravitational potential satisfying EFEs is the principal aim of all investigations in gravitational physics. This has been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. In an attempt to study the geometric and physical properties of the electromagnetic fields, different types of collineations have been investigated [12,13] along with many other interesting results. Symmetries of the energy-momentum tensor (also called matter collineations) provide conservation laws on matter fields. These enable us to know how the physical fields, occupying in certain region of spacetimes, reflect the symmetries of the metric [14].

There is a large body of recent literature which shows interest in the study of MCs [1],[15]-[22]. In a recent paper [1], the study of MCs has been taken for static spherically symmetric spacetimes (SSS) and some interesting results have been obtained. However, it was incomplete in the sense that (i) only the static case was considered and (ii) some cases were missing, in particular, for finite-dimensional MCs. In this paper, we extend the procedure to calculate MCs of SSS both for non-degenerate and also for degenerate cases with special emphasis of the metrics admitting higher symmetries and also $SO(3)$ as the maximal symmetry. We relate them with RCs and isometries.

The rest of the paper is organized as follows. The next section contains a brief review of MCs and we write down MC equations for SSS. In section 3, we shall solve these MC equations when the energy-momentum tensor is non-degenerate and in the next section MC equations are solved for the degenerate energy-momentum tensor. Section 5 contains a summary and discussion of the results obtained.

2 Matter Collineations and its Equations

Let (M, g) be a spacetime, where M is a smooth, connected, Hausdorff four-dimensional manifold and g is smooth Lorentzian metric of signature $(+ \text{ - - -})$ defined on M. The manifold M and the metric g are assumed smooth (C^∞). We shall use the usual component notation in local charts, and a covariant derivative with respect to the symmetric connection Γ associated with the metric g will be denoted by a semicolon and a partial derivative by a comma.

The geometry and matter of a spacetime are related through the EFEs given in each coordinate system of M by

$$R_{ab} - \frac{1}{2}Rg_{ab} \equiv G_{ab} = \kappa T_{ab}, \quad (a, b = 0, 1, 2, 3), \tag{1}$$

where κ is the gravitational constant, G_{ab} is the Einstein tensor, R_{ab} is the Ricci and T_{ab} is the matter (energy-momentum) tensor. Also, $R = g^{ab}R_{ab}$ is the Ricci scalar. We have assumed here that the cosmological constant $\Lambda = 0$. Using the Bianchi identities, it can easily be shown that

$$G^{ab};b = 0 \quad (\Leftrightarrow T^{ab};b = 0). \tag{2}$$

A smooth vector field ξ is said to preserve a matter symmetry [23] on M if, for each smooth local diffeomorphism ϕ_t associated ξ, the tensor T and ϕ_t^*T are equal on the domain U of ϕ_t, i.e., $T = \phi_t^*T$. Equivalently, a vector field ξ^a is said to generate a matter collineation (MC) if it satisfies the following equation

$$\pounds_\xi T_{ab} = 0 \quad \Leftrightarrow \quad \pounds_\xi G_{ab} = 0, \tag{3}$$

where $\pounds$ is the Lie derivative operator, ξ^a is the symmetry or collineation vector. Every KV is an MC but the converse is not true, in general. Collineations can be proper (non-trivial) or improper (trivial). We define a proper MC to

be an MC which is not a KV, or a homothetic vector (HV). The MC Eq.(4) can be written in component form as

$$T_{ab,c}\xi^c + T_{ac}\xi^c_{,b} + T_{cb}\xi^c_{,a} = 0. \tag{4}$$

The most general form of the metric for a spherically symmetric Lorentzian manifold is given by

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\mu(t,r)}dr^2 - e^{\lambda(t,r)}d\Omega^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The surviving components of the energy-momentum tensor, given in Appendix A, are T_{00}, T_{01} T_{11}, T_{22}, T_{33}, where $T_{33} = \sin^2\theta T_{22}$.

The MC equations can be written as follows

$$T_{00,0}\xi^0 + T_{00,1}\xi^1 + 2T_{00}\xi^0_{,0} + 2T_{01}\xi^1_{,0} = 0, \tag{6}$$

$$T_{01,0}\xi^0 + T_{01,1}\xi^1 + T_{01}\xi^0_{,0} + T_{11}\xi^1_{,0} + T_{01}\xi^1_{,1} + T_{00}\xi^0_{,1} = 0, \tag{7}$$

$$T_{00}\xi^0_{,2} + T_{01}\xi^1_{,2} + T_{22}\xi^2_{,0} = 0, \tag{8}$$

$$T_{00}\xi^0_{,3} + T_{01}\xi^1_{,3} + \sin^2\theta T_{22}\xi^3_{,0} = 0, \tag{9}$$

$$T_{11,0}\xi^0 + T_{11,1}\xi^1 + 2T_{01}\xi^0_{,1} + 2T_{11}\xi^1_{,1} = 0, \tag{10}$$

$$T_{01}\xi^0_{,2} + T_{11}\xi^1_{,2} + T_{22}\xi^2_{,1} = 0, \tag{11}$$

$$T_{01}\xi^0_{,3} + T_{11}\xi^1_{,3} + \sin^2\theta T_{22}\xi^3_{,1} = 0, \tag{12}$$

$$T_{22,0}\xi^0 + T_{22,1}\xi^1 + 2T_{22}\xi^2_{,2} = 0, \tag{13}$$

$$T_{22}\xi^2_{,3} + \sin^2\theta T_{22}\xi^3_{,2} = 0, \tag{14}$$

$$T_{22,0}\xi^0 + T_{22,1}\xi^1 + 2\cot\theta T_{22}\xi^2 + 2T_{22}\xi^3_{,3} = 0. \tag{15}$$

These are the first order non-linear partial differential equations in four variables $\xi^a(x^b)$. We solve these equations for the non-degenerate case, when

$$\det(T_{ab}) = T^2_{22}(T_{00}T_{11} - T^2_{01})\sin^2\theta \neq 0 \tag{16}$$

and for the degenerate case, where $\det(T_{ab}) = 0$. It is noticed that when $T_{01} = 0$ we shall use the notation $T_{aa} = T_a$ for the sake of brevity.

3   Matter Collineations in the Non-Degenerate Case

In this section, we shall evaluate MCs only for those cases which have non-degenerate energy-momentum tensor, i.e., $\det(T_{ab}) \neq 0$. This will be done as two cases; one when M admits higher symmetries and one when $SO(3)$ is the maximal isometry group of M. To this end, we set up the general conditions for the solution of MC equations for the non-degenerate case.

When we solve Eqs.(6)-(15) simultaneously, after some tedious algebra, we get the following solution

$$\xi^0 = \frac{T_{22}}{T_{00}T_{11} - T_{01}^2}[\{(\dot{A}_1 T_{11} - A_1' T_{01})\sin\phi - (\dot{A}_2 T_{11} - A_2' T_{01})\cos\phi\}\sin\theta$$
$$+(\dot{A}_3 T_{11} - A_3' T_{01})\cos\theta + A_4 T_{11} - A_5 T_{01}], \quad (17)$$

$$\xi^1 = \frac{-T_{22}}{T_{00}T_{11} - T_{01}^2}[\{(\dot{A}_1 T_{01} - A_1' T_{00})\sin\phi - (\dot{A}_2 T_{01} - A_2' T_{00})\cos\phi\}\sin\theta$$
$$+(\dot{A}_3 T_{01} - A_3' T_{00})\sin\theta + A_4 T_{01} - A_5 T_{00}]. \quad (18)$$

$$\xi^2 = -(A_1\sin\phi - A_2\cos\phi)\cos\theta + A_3\sin\theta$$
$$+c_1\sin\phi - c_2\cos\phi + c_4\ln(\tan\frac{\theta}{2})\sin\theta, \quad (19)$$

$$\xi^3 = -(A_1\cos\phi + A_2\sin\phi)\csc\theta + (c_1\cos\phi + c_2\sin\phi)\cot\theta + c_4\phi + c_3. \quad (20)$$

where c_1, c_2, c_3, c_4 are arbitrary constants and $A_\nu = A_\nu(t,r)$, $\nu = 1,2,3,4,5$. Here dot and prime indicate the differentiation with respect to time and r coordinate respectively. When we replace these values of ξ^a in MC Eqs.(6)-(15), we obtain the following differential constraints on A_ν with $c_4 = 0$

$$2(T_{00}T_{11} - T_{01}^2)(T_{22}\dot{A}_i) + T_{22}[(2T_{01}\dot{T}_{01} - T_{11}\dot{T}_{00} - T_{01}T_{00}')\dot{A}_i$$
$$-(2T_{00}\dot{T}_{01} - T_{01}\dot{T}_{00} - T_{00}T_{00}')A_i'] = 0, \quad (i = 1,2,3), \quad (21)$$

$$(T_{00}T_{11} - T_{01}^2)[(T_{22}\dot{A}_i)' + (T_{22}A_i')\dot{\,}] + T_{22}[(T_{01}\dot{T}_{11} - T_{11}T_{00}')\dot{A}_i$$
$$+(T_{01}T_{00}' - T_{00}\dot{T}_{11})A_i'] = 0, \quad (22)$$

$$2(T_{00}T_{11} - T_{01}^2)(T_{22}A_i')' + T_{22}[(2T_{01}T_{01}' - T_{00}T_{11}' - T_{01}\dot{T}_{11})$$
$$-(2T_{11}T_{01}' - T_{11}\dot{T}_{11} - T_{01}T_{11}')\dot{A}_i] = 0, \tag{23}$$

$$(T_{11}\dot{T}_{22} - T_{01}T_{22}')\dot{A}_i + (T_{00}T'_{22} - T_{01}\dot{T}_{22})A_i' + 2A_i = 0, \tag{24}$$

$$2(T_{00}T_{11} - T_{01}^2)(T_{22}\dot{A_4}) + T_{22}[(2T_{01}\dot{T}_{01} - T_{11}\dot{T}_{00} - T_{01}T_{00}')A_4$$
$$-(2T_{00}\dot{T}_{01} - T_{01}\dot{T}_{00} - T_{00}T_{00}')A_5] = 0, \tag{25}$$

$$(T_{00}T_{11} - T_{01}^2)[(T_{22}A_4)' + (T_{22}\dot{A_5})] + T_{22}[(T_{01}\dot{T}_{11} - T_{11}T_{00}')A_4$$
$$+(T_{01}T_{00}' - T_{00}\dot{T}_{11})A_5] = 0, \tag{26}$$

$$2(T_{00}T_{11} - T_{01}^2)(T_{22}A_5)' + T_{22}[(2T_{01}T_{01}' - T_{00}T_{11}' - T_{01}\dot{T}_{11})A_5$$
$$-(2T_{11}T_{01}' - T_{11}\dot{T}_{11} - T_{01}T_{11}')A_4] = 0, \tag{27}$$

$$(T_{11}\dot{T}_{22} - T_{01}T'_{22})A_4 + (T_{00}T'_{22} - T_{01}\dot{T}_{22})A_5 = 0. \tag{28}$$

Thus the problem of working out MCs for all possibilities of A_i, A_4, A_5 is reduced to solving the set of Eqs.(17)-(20) subject to the above constraints. We would solve these to classify MCs of the manifolds admitting higher symmetries than $SO(3)$ and $SO(3)$ as the maximal isometry group.

3.1 MCs of the Spacetimes Admitting Higher Symmetries

Here we use the constraint Eqs.(21)-(28) to evaluate MCs of the spacetimes given by Eq.(5) which admit higher symmetries than $SO(3)$. The six cases admitting symmetry groups larger than $SO(3)$ are the following:

(1) $SO(3) \otimes \mathbf{R}$, where $\mathbf{R} = \partial_t$ if and only if
(a) $\nu = \nu(r), \ \mu = \mu(r), \ \lambda = 2\ln r$ or (b) $\nu = \nu(r), \ \mu = 0, \ \lambda = 2\ln a$,
       where a is an arbitrary constant,

(2) $SO(3) \otimes \mathbf{R}$, where $\mathbf{R} = \partial_r$ if and only if
(a) $\nu = \nu(t), \ \mu = \mu(t), \ \lambda = 2\ln t$ or (b) $\nu = 0, \ \mu = \mu(t), \ \lambda = 2\ln a$,

(3) $SO(3) \otimes \mathbf{R}$, where $\mathbf{R} = \partial_t + e\partial_r$ if and only if
      $\nu = 0 = \mu$, $\lambda = \lambda(t + er)$ with $e = \pm 1$,

(4) $SO(4)$ if and only if $\nu = 0$, $\mu = 2\ln R(t)$, $\lambda = 2\ln R(t)\sin r$ such that
      $R\ddot{R} - \dot{R}^2 - 1 \neq 0$,

(5) $SO(3) \times \mathbf{R^3}$ if and only if $\nu = 0$, $\mu = 2\ln R(t)$, $\lambda = 2\ln R(t)r$ such
      that
      $R\ddot{R} - \dot{R}^2 \neq 0$,

(6) $SO(1,3)$ if and only if
(a) $\nu = 0$, $\mu = 2\ln R(t)$, $\lambda = 2\ln R(t)\sinh r$ such that
      $R\ddot{R} - \dot{R}^2 + 1 \neq 0$, or
(b) $\nu = 2\ln Q(r)$, $\mu = 0$, $\lambda = 2\ln Q(r)\cosh t$ such that
      $QQ'' - Q'^2 + 1 \neq 0$.

Case (1): In this case, we have $T_{01} = 0$ and also $\dot{T}_{ab} = 0$. Using these values,
Eqs.(17)-(28) reduce to

$$\xi^0 = \frac{T_2}{T_0}[(\dot{A}_1 \sin\phi - \dot{A}_2 \cos\phi)\sin\theta + \dot{A}_3 \cos\theta + A_4], \tag{29}$$

$$\xi^1 = \frac{T_2}{T_1}[(A_1' \sin\phi - A_2' \cos\phi)\sin\theta + A_3' \cos\theta + A_5], \tag{30}$$

$$\xi^2 = -(A_1 \sin\phi - A_2 \cos\phi)\cos\theta + A_3 \sin\theta + c_1 \sin\phi - c_2 \cos\phi, \tag{31}$$

$$\xi^3 = -(A_1 \cos\phi + A_2 \sin\phi)\csc\theta + (c_1 \cos\phi + c_2 \sin\phi)\cot\theta + c_3, \tag{32}$$

where we have used the notation $T_{aa} = T_a$ for the sake of simplicity. These
ξ^a are satisfied subject to the following differential constraints on A_ν

$$T_1\dot{A}_4 + T_0'A_5 = 0, \quad (\frac{T_2}{T_0}A_4)' + \frac{T_2}{T_0}\dot{A}_5 = 0,$$

$$(\frac{T_2}{\sqrt{T_1}}A_5)' = 0, \quad T_2'A_5 = 0, \tag{33}$$

$$2T_1\ddot{A}_i + T_0'A_i' = 0, \quad (\sqrt{\frac{T_2}{T_0}}\dot{A}_i)' = 0,$$

$$(\frac{T_2}{\sqrt{T_1}}A_i')' = 0, \quad 2T_1 A_i + T_2'A_i' = 0. \tag{34}$$

It is interesting to note that this case reduces to the non-degenerate case of the paper [1]. However, the possibility of seven MCs is recovered here which was missing there. Now the evaluation of MCs for all possibilities of A_i, A_4, A_5 is reduced to solving the set of Eqs.(29)-(32) subject to the constraints given by Eqs.(33) and (34). A complete solution of these equations is obtained by considering different possibilities of T_2. The last equation of Eq.(33) implies that either

$$(a)\ T_2' = 0, \quad or \quad (b)\ T_2' \neq 0.$$

The first case when $T_2 = \beta$, where β is an arbitrary constant, Eq.(34) gives $A_i = 0$ and consequently Eqs.(29)-(32) yield

$$\xi^0 = A_4(t,r),\ \xi^1 = A_5(t,r),$$
$$\xi^2 = c_1 \sin \phi - c_2 \cos \phi,\ \xi^3 = (c_1 \cos \phi + c_2 \sin \phi) \cot \theta + c_3. \tag{35}$$

Further, if we assume that $[\frac{T_0}{\sqrt{T_1}}(\frac{T_0'}{2T_0\sqrt{T_1}})']' \neq 0$, we obtain four MCs identical to the usual KVs of spherical symmetry given by

$$\xi = c_0 \frac{T_0}{\beta} \partial_t + c_1(\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi) + c_2(\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi) + c_3 \partial_\phi. \tag{36}$$

When $[\frac{T_0}{\sqrt{T_1}}(\frac{T_0'}{2T_0\sqrt{T_1}})']' = 0$, this implies that $\frac{T_0}{\sqrt{T_1}}[\frac{T_0'}{2T_0\sqrt{T_1}}]' = \alpha$, where α is an arbitrary constant which may be positive, zero or negative. In each case, we have six MCs.

For $\alpha > 0$, we obtain

$$
\begin{aligned}
\xi &= c_0 \frac{T_0}{\beta} \partial_t + c_1(\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi) + c_2(\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi) + c_3 \partial_\phi \\
&+ c_4(-\frac{T_0'}{2\beta\sqrt{\alpha T_1}} \sinh \sqrt{\alpha} t \partial_t + \frac{\sqrt{T_1}}{\beta} \cosh \sqrt{\alpha} t \partial_r) \\
&+ c_5(-\frac{T_0'}{2\beta\sqrt{\alpha T_1}} \cosh \sqrt{\alpha} t \partial_t + \frac{\sqrt{T_1}}{\beta} \sinh \sqrt{\alpha} t \partial_r).
\end{aligned}
\tag{37}
$$

If $\alpha = 0$, we have

$$
\begin{aligned}
\xi &= c_0 \frac{T_0}{\beta} \partial_t + c_1(\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi) + c_2(\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi) + c_3 \partial_\phi \\
&+ c_4[-\frac{T_0}{\beta}(\frac{\gamma}{2}t^2 + \frac{1}{\beta} \int \sqrt{T_1} dr) \partial_t + \frac{\sqrt{T_1}}{\beta} t \partial_r] \\
&+ c_5(\frac{T_0}{\beta} \gamma t \partial_t + \frac{\sqrt{T_1}}{\beta} \partial_r),
\end{aligned}
\tag{38}
$$

8

where $\frac{T_0'}{2T_0\sqrt{T_1}} = \gamma$, an arbitrary constant. The case $\alpha < 0$ yields

$$
\begin{aligned}
\xi \;=\;& c_0\frac{T_0}{\beta}\partial_t + c_1(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_2(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_3\partial_\phi \\
+\;& c_4(-\frac{T_0'}{2\beta\sqrt{-\alpha T_1}}\sin\sqrt{-\alpha}t\partial_t + \frac{\sqrt{T_1}}{\beta}\cos\sqrt{-\alpha}t\partial_r) \\
+\;& c_5(\frac{T_0'}{2\beta\sqrt{-\alpha T_1}}\cos\sqrt{-\alpha}t\partial_t + \frac{\sqrt{T_1}}{\beta}\sin\sqrt{-\alpha}t\partial_r).
\end{aligned}
\tag{39}
$$

In the case (b), when $T_2' \neq 0$, it follows from Eqs.(33) and (34) that for $\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 \neq 0$, we obtain the same MCs as the usual minimal KVs for spherically symmetry.

If $\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 = 0$ and $(\frac{T_2'}{\sqrt{T_0 T_1 T_2}})' \neq 0$, we have seven MCs given by

$$
\begin{aligned}
\xi \;=\;& c_0\partial_t + c_1(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_2(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_3\partial_\phi \\
+\;& c_4(-\frac{1}{\sqrt{T_1}}\sin\phi\sin\theta\partial_r - X\sin\phi\cos\theta\partial_\theta - X\cos\phi\csc\theta\partial_\phi) \\
+\;& c_5(\frac{1}{\sqrt{T_1}}\cos\phi\sin\theta\partial_r + X\cos\phi\cos\theta\partial_\theta - X\sin\phi\csc\theta\partial_\phi) \\
+\;& c_6(-\frac{1}{\sqrt{T_1}}\cos\theta\partial_r - X\sin\theta\partial_\theta),
\end{aligned}
\tag{40}
$$

where $X = \frac{T_2'}{2T_2\sqrt{T_1}}$. If we have $\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 = 0$, $(\frac{T_2'}{\sqrt{T_0 T_1 T_2}})' = 0$ and $(\frac{T_0'}{T_2'})' \neq 0$, then we get four MCs.

When $\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 = 0$, $(\frac{T_2'}{\sqrt{T_0 T_1 T_2}})' = 0$ and $\frac{T_0'}{T_2'} = \delta$, an arbitrary constant. For $\delta > 0$, we obtain

$$
\begin{aligned}
\xi \;=\;& c_0\partial_t + c_1(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_2(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_3\partial_\phi \\
+\;& c_4[(\frac{T_2}{T_0}X\sqrt{\delta}\sinh\sqrt{\delta}t\partial_t - \frac{1}{\sqrt{T_1}}\cosh\sqrt{\delta}t\partial_r)\sin\theta\sin\phi \\
-\;& (\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)X\cosh\sqrt{\delta}t] \\
+\;& c_5[(-\frac{T_2}{T_0}X\sqrt{\delta}\sinh\sqrt{\delta}t\partial_t + \frac{1}{\sqrt{T_1}}\cosh\sqrt{\delta}t\partial_r)\sin\theta\cos\phi \\
+\;& (\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)X\cosh\sqrt{\delta}t] \\
+\;& c_6[(\frac{T_2}{T_0}X\sqrt{\delta}\sinh\sqrt{\delta}t\partial_t - \frac{1}{\sqrt{T_1}}\cosh\sqrt{\delta}t\partial_r)\cos\theta + X\cosh\sqrt{\delta}t\sin\theta\partial_\theta]
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & c_7[(\frac{T_2}{T_0}X\sqrt{\delta}\cosh\sqrt{\delta}t\partial_t - \frac{1}{\sqrt{T_1}}\sinh\sqrt{\delta}t\partial_r)\sin\theta\sin\phi \\
- \quad & (\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)X\sinh\sqrt{\delta}t] \\
+ \quad & c_8[-(\frac{T_2}{T_0}X\sqrt{\delta}\cosh\sqrt{\delta}t\partial_t + \frac{1}{\sqrt{T_1}}\sinh\sqrt{\delta}t\partial_r)\sin\theta\cos\phi \\
+ \quad & (\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)X\sinh\sqrt{\delta}t \\
+ \quad & c_9(\frac{T_2}{T_0}X\sqrt{\delta}\cosh\sqrt{\delta}t\partial_t - \frac{1}{\sqrt{T_1}}X\sinh\sqrt{\delta}t\partial_r)\cos\theta + X\sinh\sqrt{\delta}t\sin\theta\partial_\theta].
\end{aligned}
$$

$$(41)$$

If $\delta = 0$, we have

$$
\begin{aligned}
\xi = \quad & c_0\partial_t + c_1(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_2(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_3\partial_\phi \\
+ \quad & c_4[(\frac{T_2}{T_0}X\partial_t - \frac{1}{\sqrt{T_1}}t\partial_r)\sin\theta\sin\phi - (\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)tX] \\
+ \quad & c_5[(-\frac{T_2}{T_0}X\partial_t + \frac{1}{\sqrt{T_1}}t\partial_r)\sin\theta\cos\phi + (\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)tX] \\
+ \quad & c_6[(\frac{T_2}{T_0}X\partial_t - \frac{1}{\sqrt{T_1}}t\partial_r)\cos\theta + tX\sin\theta] \\
+ \quad & c_7[(-\frac{1}{\sqrt{T_1}}\sin\theta\partial_r - X\cos\theta\partial_\theta)\sin\phi - X\csc\theta\cos\phi\partial_\phi] \\
+ \quad & c_8[(\frac{1}{\sqrt{T_1}}\sin\theta\partial_r + X\cos\theta\partial_\theta)\cos\phi - X\csc\theta\sin\phi\partial_\phi] \\
+ \quad & c_9(-\frac{1}{\sqrt{T_1}}\cos\theta\partial_r + X\sin\theta\partial_\theta).
\end{aligned}
$$

$$(42)$$

For $\delta < 0$, MCs are given by

$$
\begin{aligned}
\xi = \quad & c_0\partial_t + c_1(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_2(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_3\partial_\phi \\
+ \quad & c_4[(-\frac{T_2}{T_0}X\sqrt{-\delta}\sin\sqrt{-\delta}t\partial_t - \frac{1}{\sqrt{T_1}}\cos\sqrt{-\delta}t\partial_r)\sin\theta\sin\phi \\
- \quad & (\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)X\cos\sqrt{-\delta}t] \\
+ \quad & c_5[(\frac{T_2}{T_0}X\sqrt{-\delta}\sin\sqrt{-\delta}t\partial_t + \frac{1}{\sqrt{T_1}}\cos\sqrt{-\delta}t\partial_r)\sin\theta\cos\phi \\
+ \quad & (\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)X\cos\sqrt{-\delta}t] \\
+ \quad & c_6[(-\frac{T_2}{T_0}X\sqrt{-\delta}\sin\sqrt{-\delta}t\partial_t - \frac{1}{\sqrt{T_1}}\cos\sqrt{-\delta}t\partial_r)\cos\theta + X\cos\sqrt{-\delta}t\sin\theta\partial_\theta]
\end{aligned}
$$

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$$+ \quad c_7[(\frac{T_2}{T_0}X\sqrt{-\delta}\cos\sqrt{-\delta}t\partial_t - \frac{1}{\sqrt{T_1}}\sin\sqrt{-\delta}t\partial_r)\sin\theta\sin\phi$$

$$- \quad (\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)X\sin\sqrt{-\delta}t]$$

$$+ \quad c_8[-(\frac{T_2}{T_0}X\sqrt{\delta}\cos\sqrt{-\delta}t\partial_t + \frac{1}{\sqrt{T_1}}\sin\sqrt{-\delta}t\partial_r)\sin\theta\cos\phi$$

$$+ \quad (\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)X\sin\sqrt{-\delta}t$$

$$+ \quad c_9(\frac{T_2}{T_0}X\sqrt{-\delta}\cos\sqrt{-\delta}t\partial_t - \frac{1}{\sqrt{T_1}}X\sin\sqrt{-\delta}t\partial_r)\cos\theta + X\sin\sqrt{-\delta}t\sin\theta\partial_\theta].$$

$$(43)$$

From Eqs.(41)-(43), it follows that for each value of δ, we obtain ten independent MCs.

Case (2): In this case, we have $T_{01} = 0$ and $T'_a = 0$. If we use the transformations $t \leftrightarrow r$, $\xi^0 \leftrightarrow \xi^1$, $T_0 \leftrightarrow T_1$, the solution of this case can be trivially obtained as in the case (1).

Cases (4), (5), (6): The cases (4), (5) and (6a) describe Friedmann Robertson (FRW) spacetimes where as the case (6b) describes FRW like spacetimes. For these metrics, the non-vanishing components of Ricci and energy-momentum tensors are given in Appendix B. If any of T_a is zero, we get infinte dimensional MCs. For the non-degenerate case, we have $T_a \neq 0$ which implies the following possibilities

$$(a) \quad \frac{T_1}{\sqrt{T_0}}(\frac{\dot{T_2}}{2T_1\sqrt{T_0}})\dot{} - k = 0, \quad (b) \quad \frac{T_1}{\sqrt{T_0}}(\frac{\dot{T_2}}{2T_1\sqrt{T_0}})\dot{} - k \neq 0$$

with

$$(i) \quad \dot{T_1} = 0, \quad (ii) \quad \dot{T_1} \neq 0,$$

where k has the values $1, 0, -1$ according as for closed, flat and open FRW spacetimes respectively.

In the case (ai), we must have $k = 0$ and $T_1 = a \neq 0$, a is an arbitrary constant. Thus, in addition to the non-proper MCs $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}, \xi_{(4)}, \xi_{(5)}, \xi_{(6)}$ given in Appendix C, we obtain the following proper MCs

$$\xi_{(7)} \quad = \quad \frac{1}{\sqrt{T_0}}\partial_t,$$

$$\xi_{(8)} \quad = \quad r(\frac{1}{\sqrt{T_0}}\partial_t - Y\partial_r)\sin\theta\sin\phi - (\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)Y,$$

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$$\xi_{(9)} = r(\frac{1}{\sqrt{T_0}}\partial_t - Y\partial_r)\sin\theta\cos\phi - (\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)Y,$$

$$\xi_{(10)} = r(\frac{1}{\sqrt{T_0}}\partial_t - Y\partial_r)\cos\theta + Y\sin\theta\partial_\theta. \tag{44}$$

where $Y = \frac{1}{ar}\int\sqrt{T_0}dt$. This gives ten independent MCs in which six are the usual KVs of of closed FRW metric and the rest are the proper MCs.

The case (aii) also yields ten independent MCs for each value of k. For the value of $k = 1$, the proper MCs are given by

$$\xi_{(7)} = (\frac{\sqrt{T_0}}{T_1}\cot r\partial_t - Z\sin^2 r\partial_r)\csc r,$$

$$\xi_{(8)} = [(\frac{T_2}{T_0}\dot{Z}\partial_t - Z\sin r\cos r\partial_r)\sin\theta\sin\phi$$
$$- Z(\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)\csc r],$$

$$\xi_{(9)} = [(\frac{T_2}{T_0}\dot{Z}\partial_t - Z\sin r\cos r\partial_r)\sin\theta\cos\phi$$
$$- Z(\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)\csc r],$$

$$\xi_{(10)} = [(\frac{T_2}{T_0}\dot{Z}\partial_t - Z\sin r\cos r\partial_r)\cos\theta - Z\partial_\theta]\csc r. \tag{45}$$

where $Z = \frac{T_2}{2T_1\sqrt{T_0}}$. For $k = 0$, we have the following proper MCs

$$\xi_{(7)} = (\frac{1}{\sqrt{T_0}}\partial_t - rZ\partial_r),$$

$$\xi_{(8)} = [\{(\frac{rT_2}{2T_0}\dot{Z} - \frac{r}{\sqrt{T_0}})\partial_t + (\frac{r^2Z}{2} + \int\frac{\sqrt{T_0}}{T_1}dt\partial_r)\}\sin\theta\sin\phi,$$
$$- (\frac{r}{2}Z - \frac{1}{r}\int\frac{\sqrt{T_0}}{T_1}dt)(\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)],$$

$$\xi_{(9)} = [\{(\frac{rT_2}{2T_0}\dot{Z} - \frac{r}{\sqrt{T_0}})\partial_t + (\frac{r^2Z}{2} + \int\frac{\sqrt{T_0}}{T_1}dt\partial_r)\}\sin\theta\cos\phi$$
$$- (\frac{r}{2}Z - \frac{1}{r}\int\frac{\sqrt{T_0}}{T_1}dt)(\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)],$$

$$\xi_{(10)} = [\{(\frac{rT_2}{2T_0}\dot{Z} - \frac{r}{\sqrt{T_0}})\partial_t + (\frac{r^2Z}{2} + \int\frac{\sqrt{T_0}}{T_1}dt\partial_r)\}\cos\theta$$
$$- (\frac{r}{2}Z - \frac{1}{r}\int\frac{\sqrt{T_0}}{T_1}dt)\sin\theta\partial_\theta]. \tag{46}$$

For the value of $k = -1$, the four proper MCs are

$$
\begin{aligned}
\xi_{(7)} &= \frac{1}{T_1}(\sqrt{T_0}\coth r\partial_t - T_2 Z\partial_r)\csc hr, \\
\xi_{(8)} &= [(\frac{T_2}{T_0}\dot{Z}\partial_t + Z\sinh r\cosh r\partial_r)\sin\theta\sin\phi \\
&\quad - Z(\cos\theta\sin\phi\partial_\theta + \csc\theta\cos\phi\partial_\phi)]\csc hr, \\
\xi_{(9)} &= [(\frac{T_2}{T_0}\dot{Z}\partial_t + Z\sinh r\cosh r\partial_r)\sin\theta\cos\phi \\
&\quad - Z(\cos\theta\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\phi)]\csc hr, \\
\xi_{(10)} &= [(\frac{T_2}{T_0}\dot{Z}\partial_t + Z\sinh r\cosh r\partial_r)\cos\theta - Z\partial_\theta]\csc hr. \qquad (47)
\end{aligned}
$$

Thus we obtain ten independent MCs for each value of k in which six are the usual isometries of FRW metric and the the remaining four are the proper MCs.

For the case (bi), we must require that $k \neq 0$. When $k = 1$, we obtain one proper MC given by

$$
\xi_{(7)} = \frac{T_2}{\sqrt{T_0}a}\csc^2 r\partial_t. \qquad (48)
$$

For $k = -1$, proper MC is

$$
\xi_{(7)} = \frac{T_2}{\sqrt{T_0}a}\csc h^2 r\partial_t. \qquad (49)
$$

This case gives seven independent MCs in which six are non-proper and one is proper MC. It can be checked that the case (bii) gives six independent MCs for each value of k which are usual KVs of FRW spacetimes. Similarly, the case (6b) can be solved to give either six, seven or ten MCs.

3.2 MCs of the Spacetimes Admitting $SO(3)$ as the Maximal Isometry Group

In this section, we evaluate MCs of the spherically symmetric spacetimes which admit $SO(3)$ as the maximal isometry group. In these solutions, we take any additional MC (if exists) be orthogonal to the $SO(3)$ orbit. For this we must require that $A_i \equiv 0$ and consequently, it follows from Eqs.(17)-(20)

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that $\xi^0 = \xi^0(t,r)$, $\xi^1 = \xi^1(t,r)$, $\xi^2 = 0$, $\xi^3 = 0$. It is mentioned here that we are considering only diagonal metrics for this case. The non-diagonal metrics can be solved in a similar way. If we make use of the following substitutions

$$\frac{T_2}{T_0}A_4 = C(t,r), \ \frac{T_2}{T_1}A_5 = D(t,r), \ \sqrt{T_0} = A(t,r), \ \sqrt{T_1} = B(t,r)$$

in the constraint Eqs.(21)-(28), then it follows that

$$\dot{C} = -\dot{A}C - A'D, \tag{50}$$

$$A^2 C' + B^2 \dot{D} = 0, \tag{51}$$

$$D' = -\dot{B}C - B'D, \tag{52}$$

$$\dot{T}_2 C + T_2' D = 0. \tag{53}$$

To solve this system of equations, we have the following possibilities:
(i) $\dot{T}_2 = 0$, $T_2' \neq 0$,   (ii) $\dot{T}_2 \neq 0$, $T_2' = 0$,
(iii) $\dot{T}_2 \neq 0$, $T_2' \neq 0$,   (iv) $\dot{T}_2 = 0$, $T_2' = 0$.

    The first possibility does not provide any proper MC if we assume that $\dot{T}_1 \neq 0$. However, the assumption $\dot{T}_1 = 0$, $\dot{T}_0 \neq 0$ gives infinite dimensional MCs.

    The second case shows that there does not exist a proper MC with the constraint $T_0' \neq 0$ but the constraints $T_0' = 0$, $T_1' \neq 0$ provide infinite dimensional MCs.

    In the third case, when

$$T_0 T_2'[\frac{T_1'\dot{T}_2 - \dot{T}_1 T_2'}{2\sqrt{T_1}\dot{T}_2} + \{\ln(\frac{\dot{T}_2}{T_2'})\}'] + T_1\dot{T}_2[\frac{T_0'\dot{T}_2 - \dot{T}_0 T_2'}{2\sqrt{T_0}T_2'} + \{\ln(\frac{\dot{T}_2}{T_2'})\dot{\}}] \neq 0,$$

we do not have a proper MC. However, if

$$T_0 T_2'[\frac{T_1'\dot{T}_2 - \dot{T}_1 T_2'}{2\sqrt{T_1}\dot{T}_2} + \{\ln(\frac{\dot{T}_2}{T_2'})\}'] + T_1\dot{T}_2[\frac{T_0'\dot{T}_2 - \dot{T}_0 T_2'}{2\sqrt{T_0}T_2'} + \{\ln(\frac{\dot{T}_2}{T_2'})\dot{\}}] = 0.$$

and

$$(\frac{T_0'\dot{T}_2 - \dot{T}_0 T_2'}{2\sqrt{T_0}\dot{T}_2})' = [\frac{T_2'}{\dot{T}_2}\{\frac{\dot{T}_1 T_2' - T_1'\dot{T}_2}{2\sqrt{T_1}T_2'} - (\frac{\dot{T}_2}{T_2'})'\}]$$

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then there exists a proper MC given by

$$\exp(\int \frac{T_0'\dot{T}_2 - \dot{T}_0 T_2'}{2\sqrt{T_0 \dot{T}_2}}dt)(\partial_t - \frac{\dot{T}_2}{T_2'}\partial_r). \tag{54}$$

If

$$(\frac{T_0'\dot{T}_2 - \dot{T}_0 T_2'}{2\sqrt{T_0 \dot{T}_2}})' \neq [\frac{T_2'}{\dot{T}_2}\{\frac{\dot{T}_1 T_2' - T_1' \dot{T}_2}{2\sqrt{T_1 T_2'}} - (\frac{\dot{T}_2}{T_2'})'\}]$$

then this case gives infinite number of MCs.

In the last case, we solve Eqs.(50)-(52) which imply that $\dot{A}B' - A'\dot{B} \equiv \psi(t, r)$. If $\psi = 0$, then we must have $\dot{C} = 0 = D'$ for a non-trivial solution. Thus the constraints $\dot{T}_0 T_1' - T_0' \dot{T}_1 = 0$ together with

$$T_0' \neq 0, \ [\frac{T_1}{T_0}(\frac{\dot{T}_1}{T_0'})\dot{}] = 0,$$

yield the following proper MC

$$\exp(\int \frac{T_1}{T_0}(\frac{\dot{T}_0}{T_0'})\dot{}dr)(\partial_t - \frac{\dot{T}_0}{T_0'}\partial_r). \tag{55}$$

However, for $\dot{T}_0 T_1' - T_0' \dot{T}_1 = 0$ together with

$$T_0' = 0, \ T_1' \neq 0, \ [\frac{T_1}{T_0}(\frac{\dot{T}_1}{T_1'})\dot{}] = 0,$$

we obtain the proper MC given by

$$\exp(\int \frac{T_1}{T_0}(\frac{\dot{T}_1}{T_1'})\dot{}dr)(\partial_t - \frac{\dot{T}_1}{T_1'}\partial_r). \tag{56}$$

The constraint $\dot{T}_0 T_1' - T_0' \dot{T}_1 = 0$ along with $T_0' = 0, \ T_1' = 0 = \dot{T}_1, \ \dot{T}_0 \neq 0$ gives infinite many MCs.

For $\psi \neq 0$, we must have $\dot{C} \neq 0, D' \neq 0$ for a non-trivial solution. Let us express $\dot{C}$ and D' as E and F respectively so that

$$C = -\frac{B'}{\psi}E(t, r) + \frac{A'}{\psi}F(t, r), \tag{57}$$

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$$D = \frac{\dot{B}}{\psi}E(t,r) - \frac{\dot{A}}{\psi}F(t,r). \tag{58}$$

We obtain two linearly independent MCs which are orthogonal to $T_e(SO(3))$ and are given by

$$X_1 = \frac{E}{\psi}(-B'\partial_t + \dot{B}\partial_r), \tag{59}$$

$$X_2 = \frac{F}{\psi}(A'\partial_t - \dot{A}\partial_r). \tag{60}$$

The Lie bracket of these vector fields is

$$[X_1, X_2] = \frac{Ft(\dot{A}E' - A'\dot{E})}{E\psi}X_1 + \frac{E(\dot{B}F' - B'\dot{F})}{F\psi}X_2. \tag{61}$$

For its closedness, we must have $\frac{F(\dot{A}E' - A'\dot{E})}{E\psi} = a_1$ and $\frac{E(\dot{B}F' - B'\dot{F})}{F\psi} = a_2$, where a_1 and a_2 are constants. From here we have either (i) $a_1 \neq 0$, $a_2 = 0$, or (ii) $a_1 = 0$, $a_2 \neq 0$ or (iii) $a_1 = 0 = a_2$. The first two possibilities contradict the assumption that $\psi \neq 0$. This shows that the third possibility closes the Lie algebra. Thus we have

$$C' = \frac{B^2}{A^2 A'^2 + B^2 \dot{A}^2}[\{A'(\ddot{A} - \dot{A}^2) - \dot{A}(\dot{A}' - A'\dot{B})\}C$$
$$+ \{A'(\dot{A}' - \dot{A}A') - \dot{A}(A'' - A'B')\}D], \tag{62}$$

$$\dot{D} = -\frac{A^2}{A^2 A'^2 + B^2 \dot{A}^2}[\{A'(\ddot{A} - \dot{A}^2) - \dot{A}(\dot{A}' - A'\dot{B})\}C$$
$$+ \{A'(\dot{A}' - \dot{A}A') - \dot{A}(A'' - A'B')\}D], \tag{63}$$

along with the compatibility constraint in the components T_0 and T_1 of the energy-momentum tensor given by

$$(\ln \frac{A'}{\dot{A}}e^{A-B})'(\ln \frac{\dot{B}}{B'}e^{B-A})\dot{} - (\ln \frac{A'}{\dot{A}}e^{A-B})\dot{}(\ln \frac{\dot{B}}{B'}e^{B-A})' = 0. \tag{64}$$

4 Matter Collineations in the Degenerate Case

In this section only those cases will be considered for which the energy-momentum tensor is degenerate, i.e., $\det(T_{ab}) = 0$.

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4.1 MCs of the Manifolds Admitting Higher Symmetries

Here we would discuss the MCs of the manifolds admitting higher symmetries than $SO(3)$. For higher symmetries, all metrics have $T_{01} = 0$ except the case (3) of the last section. Thus we would discuss the spacetimes for which $T_{01} = 0$ and $\det(T_{ab}) = 0$, i.e., when at least one of the T_a or their combination is zero. It can be shown that for $T_1 = 0$, $T_k \neq 0$, $k = 0, 2$ (case (1) of section 3), we obtain infinite dimensional MCs. The solution for $T_0 = 0$, $T_l \neq 0$, $l = 1, 2$ (case (2) of section 3) also gives infinite dimensional MCs. These have been discussed in detail elsewhere [1]. Here we are interested in exploring the possibilities of finite MCs.

When $T_k' \neq 0$, $(\frac{T_0}{T_2})' \neq 0$, we obtain four MCs which are the usual KVs of the spherical symmetry. For $T_k' \neq 0$, $(\frac{T_0}{T_2})' = 0$, we obtain ten independent MCs. These are

$$\xi^0 = \beta[(\dot{g}_1 \sin \phi - \dot{g}_2 \cos \phi) \sin \theta + \dot{g}_3 \cos \theta] + c_0, \quad \xi^1 = 0,$$
$$\xi^2 = -(g_1 \sin \phi - g_2 \cos \phi) \cos \theta + g_3 \sin \theta + c_1 \sin \phi - c_2 \cos \phi,$$
$$\xi^3 = -(g_1 \cos \phi + g_2 \sin \phi) \csc \theta + (c_1 \cos \phi + c_2 \sin \phi) \cot \theta + c_3, \quad (65)$$

where $\beta = \frac{T_2}{T_0} \neq 0$, is an arbitrary constant and the function g satisfies the following constraint

$$\beta \ddot{g}_i(t) - g_i(t) = 0. \quad (66)$$

The solution for the non-static case can be obtained trivially which turn out to be the same with different constraints.

4.2 MCs of the Manifolds Admitting $SO(3)$ as the Maximal Isometry Group

The metrics which admit $SO(3)$ as the maximal symmetry group yield $\xi^2 = 0 = \xi^3$ and the MC equations reduce to six independent equations which involve the following equation

$$\dot{T}_{22}\xi^0 + T_{22}'\xi^1 = 0. \quad (67)$$

This gives rise to the following four cases:
(i) $\dot{T}_{22} = 0$, $T_{22}' \neq 0$,   (ii) $\dot{T}_{22} \neq 0$, $T_{22}' = 0$,
(iii) $\dot{T}_{22} \neq 0$, $T_{22}' \neq 0$,   (iv) $\dot{T}_{22} = 0$, $T_{22}' = 0$.

If we solve these cases, we may have interesting physical consequences. These will be discussed somewhere else.

5   Conclusion

In this paper, we have attempted to classify the most general spherically symmetric spacetimes according to their MCs. We have found a general solution of the MC equations for the non-degenerate, diagonal and non-diagonal energy-momentum tensor. Further, we have classified spacetimes admitting higher symmetries than $SO(3)$ and those which admit $SO(3)$ as the maximal isometry group for both non-degenerate and degenerate cases. It is found that for the non-degenerate and degenerate cases, we recover the earlier known results [1] as a special case. We also obtain some interesting missing results in the earlier work. It is mentioned here that MCs found here coincide with RCs but the constraints are entirely different. The summary of the results can be given below in the form of tables.

Table 1. MCs of Case (1) for the Non-degenerate Case admitting Higher Symmetries

Cases	MCs	Constraints
1ai	4	$[\frac{T_0}{\sqrt{T_1}}(\frac{T_0'}{2T_0\sqrt{T_1}})']' \neq 0$
1aii	6	$[\frac{T_0}{\sqrt{T_1}}(\frac{T_0'}{2T_0\sqrt{T_1}})']' = 0$
1bi	4	$\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 \neq 0$
1bii	7	$\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 = 0$, $(\frac{T_2'}{\sqrt{T_0T_1T_2}})' \neq 0$
1biii	4	$\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 = 0$, $(\frac{T_2'}{\sqrt{T_0T_1T_2}})' \neq 0$, $(\frac{T_0'}{T_2})' \neq 0$
1biv	10	$\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 = 0$, $(\frac{T_2'}{\sqrt{T_0T_1T_2}})' \neq 0$, $(\frac{T_0'}{T_2})' = 0$

Notice that MCs for the case (2) are the same as for the case (1) which can be obtained trivially by using the transformations given in the section (3).

Table 2. MCs of Cases (4),(5),(6) for the Non-degenerate Case admitting Higher Symmetries.

Cases	MCs	Constraints
4ai	10	$\frac{T_1}{\sqrt{T_0}}\left(\frac{T_2}{2T_1\sqrt{T_0}}\right)^{\cdot} - k = 0,\ \dot{T_1} = 0$
4aii	10	$\frac{T_1}{\sqrt{T_0}}\left(\frac{T_2}{2T_1\sqrt{T_0}}\right)^{\cdot} - k = 0,\ \dot{T_1} \neq 0$
4bi	7	$\frac{T_1}{\sqrt{T_0}}\left(\frac{T_2}{2T_1\sqrt{T_0}}\right)^{\cdot} - k \neq 0,\ \dot{T_1} = 0$
4bii	6	$\frac{T_1}{\sqrt{T_0}}\left(\frac{T_2}{2T_1\sqrt{T_0}}\right)^{\cdot} - k \neq 0,\ \dot{T_1} \neq 0$

It is noted here that the cases (4), (5) and (6a) describes FRW metrics and (6b) FRW like metrics and have the same MCS in all cases as given above.

Table 3. MCs for the Non-degenerate Case admitting $SO(3)$ as the Maximal Symmetry.

Cases	MCs	Constraints
ia	No Proper	$\dot{T_2} = 0,\ T_2' \neq 0,\ \dot{T_1} \neq 0$
ib	Infinite No. of MCs	$\dot{T_2} = 0,\ T_2' \neq 0,\ \dot{T_1} = 0,\ \dot{T_0} \neq 0$
iia	No Proper	$\dot{T_2} \neq 0,\ T_2' = 0,\ T_0' \neq 0$
iib	Infinite No. of MCs	$\dot{T_2} \neq 0,\ T_2' = 0,\ T_0' = 0,\ T_1' \neq 0$
iiia	No Proper	$\dot{T_2} \neq 0,\ T_2' \neq 0,\ T_0 T_2'[\frac{T_1'\dot{T_2}-\dot{T_1}T_2'}{2\sqrt{T_1}\dot{T_2}} + \{\ln(\frac{\dot{T_2}}{T_2'})\}'] + T_1 \dot{T_2}[\frac{T_0'\dot{T_2}-\dot{T_0}T_2'}{2\sqrt{T_0}T_2'} + \{\ln(\frac{\dot{T_2}}{T_2'})\}^{\cdot}] \neq 0$
iiib	One Proper	$\dot{T_2} \neq 0,\ T_2' \neq 0,\ T_0 T_2'[\frac{T_1'\dot{T_2}-\dot{T_1}T_2'}{2\sqrt{T_1}\dot{T_2}} + \{\ln(\frac{\dot{T_2}}{T_2'})\}'] + T_1 \dot{T_2}[\frac{T_0'\dot{T_2}-\dot{T_0}T_2'}{2\sqrt{T_0}T_2'} + \{\ln(\frac{\dot{T_2}}{T_2'})\}^{\cdot}] = 0,\ (\frac{T_0'\dot{T_2}-\dot{T_0}T_2'}{2\sqrt{T_0}\dot{T_2}})' = [\frac{T_2'}{\dot{T_2}}\{\frac{\dot{T_1}T_2'-T_1'\dot{T_2}}{2\sqrt{T_1}T_2'} - (\frac{\dot{T_2}}{T_2'})'\}^{\cdot}]$
iiic	Infinite No. of MCs	$\dot{T_2} \neq 0,\ T_2' \neq 0,\ T_0 T_2'[\frac{T_1'\dot{T_2}-\dot{T_1}T_2'}{2\sqrt{T_1}\dot{T_2}} + \{\ln(\frac{\dot{T_2}}{T_2'})\}'] + T_1 \dot{T_2}[\frac{T_0'\dot{T_2}-\dot{T_0}T_2'}{2\sqrt{T_0}T_2'} + \{\ln(\frac{\dot{T_2}}{T_2'})\}^{\cdot}] = 0,\ (\frac{T_0'\dot{T_2}-\dot{T_0}T_2'}{2\sqrt{T_0}\dot{T_2}})' \neq [\frac{T_2'}{\dot{T_2}}\{\frac{\dot{T_1}T_2'-T_1'\dot{T_2}}{2\sqrt{T_1}T_2'} - (\frac{\dot{T_2}}{T_2'})'\}^{\cdot}]$
iva	One Proper	$\dot{T_2} = 0,\ T_2' = 0,\ \dot{T_0}T_1' - T_0'\dot{T_1} = 0,\ T_0' \neq 0,\ [\frac{T_1}{T_0}(\frac{\dot{T_1}}{T_0'})^{\cdot}] = 0$
ivb	One Proper	$\dot{T_2} = 0,\ T_2' = 0,\ \dot{T_0}T_1' - T_0'\dot{T_1} = 0,\ T_0' = 0,\ T_1' \neq 0,\ [\frac{T_1}{T_0}(\frac{\dot{T_1}}{T_0'})^{\cdot}] = 0$
ivc	Infinite No. MCs	$\dot{T_2} = 0,\ T_2' = 0,\ \dot{T_0}T_1' - T_0'\dot{T_1} = 0,\ T_0' = 0,\ T_1' = 0 = \dot{T_1},\ \dot{T_0} \neq 0$

Table 4. MCs of Degenerate Case admitting Higher Symmetries.

Cases	MCs	Constraints
*	4	$T_k' \neq 0$, $(\frac{T_0}{T_1})' \neq 0$
**	10	$T_k' \neq 0$, $(\frac{T_0}{T_1})' = 0$

It can be seen from the above tables that each case has different constraints on the energy-momentum tensor. It would be interesting to solve these constraints or at least examples should be constructed to check the dimensions of the MCs. We are able to classify MCs of the spacetimes with $SO(3)$ as the maximal isometry group only for the non-degenerate case. However, it needs to be completed for the degenerate case. Also, the case (3) of the section (3) admitting higher symmetries is kept open. These would be discussed in a separate work.

Appendix A

The surviving components of the Ricci tensor are

$$R_{00} = \frac{1}{4}e^{\nu-\mu}(2\nu'' + \nu'^2 - \nu'\mu' + 2\nu'\lambda') - \frac{1}{4}(2\ddot{\mu} + \dot{\mu}^2 - \dot{\nu}\dot{\mu} + 4\ddot{\lambda} + 2\dot{\lambda}^2 - 2\dot{\nu}\dot{\lambda}),$$

$$R_{01} = -\frac{1}{2}(2\dot{\lambda}' + \dot{\lambda}\lambda' - \nu'\dot{\lambda} - \dot{\mu}\lambda'),$$

$$R_{11} = \frac{1}{4}e^{\mu-\nu}(2\ddot{\mu} + \dot{\mu}^2 - \dot{\nu}\dot{\mu} + 2\dot{\mu}\dot{\lambda}) - \frac{1}{4}(2\nu'' + \nu'^2 - \nu'\mu' + 4\lambda'' + 2\lambda'^2 - 2\mu'\lambda'),$$

$$R_{22} = \frac{1}{4}e^{\lambda-\nu}(2\ddot{\lambda} + 2\dot{\lambda}^2 - \dot{\nu}\dot{\lambda} + \dot{\mu}\dot{\lambda}) - \frac{1}{4}e^{\lambda-\mu}(2\lambda'' + 2\lambda'^2 - \mu'\lambda' + \nu'\lambda') + 1,$$

$$R_{33} = R_{22}\sin^2\theta. \tag{A1}$$

The Ricci scalar is given by

$$
\begin{aligned}
R ={}& \frac{1}{2}e^{-\mu}(2\nu'' + \nu'^2 - \nu'\mu' + 2\nu'\lambda' - 2\mu'\lambda' + 3\lambda'^2 + 4\lambda'') - 2e^{-\lambda} \\
&- \frac{1}{2}e^{-\nu}(2\ddot{\mu} + \dot{\mu}^2 - \dot{\nu}\dot{\mu} - 2\dot{\nu}\dot{\lambda} + 2\dot{\mu}\dot{\lambda} + 3\dot{\lambda}^2 + 4\ddot{\lambda}).
\end{aligned}
\tag{A2}
$$

Using Einstein field equations (1), the non-vanishing components of energy-momentum tensor T_{ab} are

$$T_{00} = \frac{1}{4}(\dot{\lambda}^2 + 2\dot{\mu}\dot{\lambda}) - \frac{1}{4}e^{\nu-\mu}(4\lambda'' + 3\lambda'^2 - 2\mu'\lambda') + e^{\nu-\lambda}, \quad T_{01} = R_{01},$$

$$T_{11} = \frac{1}{4}(\lambda'^2 + 2\nu'\lambda') - \frac{1}{4}e^{\mu-\nu}(4\ddot{\lambda} + 3\dot{\lambda}^2 - 2\dot{\nu}\dot{\lambda}) - e^{\mu-\lambda},$$

$$
\begin{aligned}
T_{22} ={}& \frac{1}{4}e^{\lambda-\mu}(2\nu'' + \nu'^2 - \nu'\mu' + \nu'\lambda' - \mu'\lambda' + \lambda'^2 + 2\lambda'') \\
&- \frac{1}{4}e^{\lambda-\nu}(2\ddot{\mu} + \dot{\mu}^2 - \dot{\nu}\dot{\mu} - \dot{\nu}\dot{\lambda} + \dot{\mu}\dot{\lambda} + \dot{\lambda}^2 + 2\ddot{\lambda}),
\end{aligned}
$$

$$T_{33} = T_{22}\sin^2\theta. \tag{A3}$$

Appendix B

The non-vanishing components of the Ricci tensor for FRW spacetimes are given by

$$R_0 = -3\frac{\ddot{R}}{R},$$

$$R_1 = \frac{(R^3)^{\ddot{}}}{3R} - 2k,$$
$$R_2 = R_1 \Sigma^2(k, r),$$
$$R_3 = R_2 \sin^2 \theta, \tag{B1}$$

where

$$\Sigma(k, r) = \sin r, \quad for \quad k = 1,$$
$$= r, \quad for \quad k = 0,$$
$$= \sinh r, \quad for \quad k = -1.$$

The Ricci scalar is given by

$$R = -\frac{6}{R^2}(R\ddot{R} + \dot{R}^2 - k). \tag{B2}$$

Now, the surviving components of energy-momentum tensor for FRW spacetimes are given by

$$T_0 = \frac{3}{R^2}(\dot{R}^2 - k),$$
$$T_1 = -(2R\ddot{R} + \dot{R}^2) + k,$$
$$T_2 = T_1 \Sigma^2(k, r),$$
$$T_3 = T_2 \sin^2 \theta. \tag{B3}$$

Appendix C

Linearly independent KVs associated with the FRW spacetimes are given by [24] for $k = 1$

$$\xi_{(1)} = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi,$$
$$\xi_{(2)} = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi,$$
$$\xi_{(3)} = \partial_\phi$$
$$\xi_{(4)} = (\sin\theta\partial_r + \cot r\cos\theta\partial_\theta)\sin\phi + \cot r\csc\theta\cos\phi\partial_\phi,$$
$$\xi_{(5)} = (\sin\theta\partial_r + \cot r\cos\theta\partial_\theta)\cos\phi - \cot r\csc\theta\sin\phi\partial_\phi,$$
$$\xi_{(6)} = \cos\theta\partial_r - \cot r\sin\theta\partial_\theta. \tag{C1}$$

For $k = 0$, we have

$$
\begin{aligned}
\xi_{(1)} &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \\
\xi_{(2)} &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \\
\xi_{(3)} &= \partial_\phi \\
\xi_{(4)} &= (\sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta)\sin\phi + \frac{1}{r}\csc\theta\cos\phi\partial_\phi, \\
\xi_{(5)} &= (\sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta)\cos\phi - \frac{1}{r}\csc\theta\sin\phi\partial_\phi \\
\xi_{(6)} &= (\cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta).
\end{aligned}
\tag{C2}
$$

For $k = -1$

$$
\begin{aligned}
\xi_{(1)} &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \\
\xi_{(2)} &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \\
\xi_{(3)} &= \partial_\phi \\
\xi_{(4)} &= (\sin\theta\partial_r + \coth r\cos\theta\partial_\theta)\sin\phi + \coth r\csc\theta\cos\phi\partial_\phi, \\
\xi_{(5)} &= (\sin\theta\partial_r + \coth r\cos\theta\partial_\theta)\cos\phi - \coth r\csc\theta\sin\phi\partial_\phi, \\
\xi_{(6)} &= \cos\theta\partial_r - \coth r\sin\theta\partial_\theta.
\end{aligned}
\tag{C3}
$$

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